LPENSL-TH-05

Exact results for the σ^z two-point function of the XXZ chain at $\Delta = 1/2$

N. Kitanine[1], **J. M. Maillet**[2], **N. A. Slavnov**[3], **V. Terras**[4]

Abstract

We propose a new multiple integral representation for the correlation function $\langle \sigma_1^z \sigma_{m+1}^z \rangle$ of the XXZ spin-$\frac{1}{2}$ Heisenberg chain in the disordered regime. We show that for $\Delta = 1/2$ the integrals can be separated and computed exactly. As an example we give the explicit results up to the lattice distance $m = 8$. It turns out that the answer is given as integer numbers divided by $2^{(m+1)^2}$.

[1]LPTM, UMR 8089 du CNRS, Université de Cergy-Pontoise, France, kitanine@ptm.u-cergy.fr
[2]Laboratoire de Physique, UMR 5672 du CNRS, ENS Lyon, France, maillet@ens-lyon.fr
[3]Steklov Mathematical Institute, Moscow, Russia, nslavnov@mi.ras.ru
[4]LPTA, UMR 5207 du CNRS, Montpellier, France, terras@lpta.univ-montp2.fr

In this paper we study the correlation functions of the XXZ spin-$\frac{1}{2}$ Heisenberg chain [1]. The Hamiltonian of this model is given by

$$H = \sum_{m=1}^{M} \left\{ \sigma_m^x \sigma_{m+1}^x + \sigma_m^y \sigma_{m+1}^y + \Delta(\sigma_m^z \sigma_{m+1}^z - 1) \right\}, \tag{1}$$

where Δ is the anisotropy parameter and $\sigma_m^{x,y,z}$ are local spin operators associated with the m-th site of the chain. The boundary conditions are periodical.

In 2001 Razumov and Stroganov [2] observed that, at the special value of the anisotropy parameter $\Delta = 1/2$, the model exhibits a series of remarkable properties. It was conjectured in particular that, in the thermodynamic limit, the probability to find a ferromagnetic string of length m in the antiferromagnetic ground state is proportional to the number of alternating sign matrices of size $m \times m$. This conjecture was proved in [3] using multiple integral representations of the correlation functions [4], [5], [6], [7]. We use here the same approach to study the two-point correlation function of the third components of local spin.

Following the strategy of [8], [7], we introduce the operator

$$Q_\kappa(m) = \prod_{n=1}^{m} \left(\frac{1+\kappa}{2} + \frac{1-\kappa}{2} \cdot \sigma_n^z \right), \tag{2}$$

where κ is a complex number. The ground state expectation value of this operator is the generating function for the correlation functions of the third components of spin, namely,

$$\langle \sigma_1^z \sigma_{m+1}^z \rangle = 2\, \frac{\partial^2}{\partial \kappa^2} D_m^2 \langle Q_\kappa(m) \rangle \bigg|_{\kappa=1} - 1, \tag{3}$$

where D_m^2 means the second lattice derivative.

To study the generating function $\langle Q_\kappa(m) \rangle$ in the thermodynamic limit, we use the following multiple integral representation:

$$\begin{aligned}\langle Q_\kappa(m) \rangle = \sum_{n=0}^{m} \frac{\kappa^{m-n}}{n!(m-n)!} \oint\limits_{\Gamma\{-i\zeta/2\}} \frac{d^m z}{(2\pi i)^m} \int\limits_{\mathbb{R}-i\zeta} d^n\lambda \int\limits_{\mathbb{R}} d^{m-n}\lambda \cdot \prod_{j=1}^{m} \varphi^m(z_j) \varphi^{-m}(\lambda_j) \\ \times \prod_{j=1}^{n} \left\{ t(z_j, \lambda_j) \prod_{k=1}^{m} \frac{\sinh(z_j - \lambda_k - i\zeta)}{\sinh(z_j - z_k - i\zeta)} \right\} \prod_{j=n+1}^{m} \left\{ t(\lambda_j, z_j) \prod_{k=1}^{m} \frac{\sinh(\lambda_k - z_j - i\zeta)}{\sinh(z_k - z_j - i\zeta)} \right\} \\ \times \prod_{j=1}^{m} \prod_{k=1}^{m} \frac{\sinh(\lambda_k - z_j - i\zeta)}{\sinh(\lambda_k - \lambda_j - i\zeta)} \cdot \det_m \left(\frac{i}{2\zeta \sinh \frac{\pi}{\zeta}(\lambda - z)} \right).\end{aligned} \tag{4}$$

Here,

$$\Delta = \cos\zeta, \qquad t(z,\lambda) = \frac{-i\sin\zeta}{\sinh(z-\lambda)\sinh(z-\lambda-i\zeta)}, \qquad \varphi(z) = \frac{\sinh(z - i\frac{\zeta}{2})}{\sinh(z + i\frac{\zeta}{2})}, \tag{5}$$

and the integrals over the variables z_j are taken with respect to a closed contour Γ which surrounds the point $-i\zeta/2$ and does not contain any other singularities of the integrand.

The equation (4) can be derived from the one obtained for $\langle Q_\kappa(m)\rangle$ in [7] by expanding the integrand into power series over κ with forthcoming moving a part of the integration contours for variables λ_j to the lower half-plane.

The equation (4) is valid for the homogeneous XXZ chain with arbitrary $-1<\Delta<1$. If we consider the inhomogeneous XXZ model with inhomogeneities $\xi_1,\dots,\xi_m$, then one should replace in the representation (4) the function φ^m in the following way:

$$\varphi^m(z)\to\prod_{b=1}^{m}\frac{\sinh(z-\xi_b-i\zeta)}{\sinh(z-\xi_b)},\qquad \varphi^{-m}(\lambda)\to\prod_{b=1}^{m}\frac{\sinh(\lambda-\xi_b)}{\sinh(\lambda-\xi_b-i\zeta)}. \tag{6}$$

In order to come back to the homogeneous case, one should set $\xi_k=-i\zeta/2$, $k=1,\dots,m$ in (6).

In the inhomogeneous model, the integration contour Γ surrounds the points $\xi_1,\dots,\xi_m$, and the integrals over z_j are therefore equal to the sum of the residues of the integrand in these simple poles. Computing these integrals and setting $\zeta=\pi/3$, we obtain an alternating sum over the permutations of the set $\{\xi_1,\dots,\xi_m\}$:

$$\begin{aligned}\langle Q_\kappa(m)\rangle=&\frac{(-1)^{\frac{m^2-m}{2}}}{m!}\left(\frac{\sqrt{3}}{2\pi}\right)^m 2^{-m^2}\prod_{a>b}^{m}\frac{\sinh 3(\xi_a-\xi_b)}{\sinh^2(\xi_a-\xi_b)}\\&\times\sum_{n=0}^{m}\kappa^{m-n}C_m^n\sum_p(-1)^{[p]}\prod_{a=1}^{m}\prod_{b=n+1}^{m}\frac{\sinh(\xi_{p(b)}-\xi_a-\frac{i\pi}{3})}{\sinh(\xi_{p(b)}-\xi_a+\frac{i\pi}{3})}\\&\int\limits_{\mathbb{R}-i\zeta}d^n\lambda\int\limits_{\mathbb{R}}d^{m-n}\lambda\cdot\frac{\prod\limits_{a>b}^{m}\sinh(\lambda_a-\lambda_b)\prod\limits_{j=1}^{n}t(\xi_{p(j)},\lambda_j)\prod\limits_{j=n+1}^{m}t(\lambda_j,\xi_{p(j)})}{\prod\limits_{a=1}^{m}\left\{\prod\limits_{b=1}^{n}\sinh(\lambda_a-\xi_{p(b)}-\frac{i\pi}{3})\prod\limits_{b=n+1}^{m}\sinh(\lambda_a-\xi_{p(b)}+\frac{i\pi}{3})\right\}}.\end{aligned} \tag{7}$$

It is easy to see that the double products containing the parameters $\{\lambda\}$ can be written as a Cauchy determinant with $\sinh^{-1}(\lambda_j-\xi_{p(k)}-\frac{i\pi}{3})$ in the first n columns and $\sinh^{-1}(\lambda_j-\xi_{p(k)}+\frac{i\pi}{3})$ in the remaining ones . Thus, the integrand can be presented as a determinant of a block-matrix:

$$\begin{aligned}\langle Q_\kappa(m)\rangle=&\frac{1}{2^{m^2}m!}\left(\frac{\sqrt{3}}{2\pi}\right)^m\prod_{a>b}^{m}\frac{\sinh 3(\xi_a-\xi_b)}{\sinh^3(\xi_a-\xi_b)}\\&\times\sum_{n=0}^{m}\kappa^{m-n}C_m^n\sum_p\prod_{a=1}^{n}\prod_{b=n+1}^{m}\frac{\sinh(\xi_{p(b)}-\xi_{p(a)}-\frac{i\pi}{3})\sinh(\xi_{p(a)}-\xi_{p(b)})}{\sinh^2(\xi_{p(b)}-\xi_{p(a)}+\frac{i\pi}{3})}\\&\int\limits_{\mathbb{R}-i\zeta}d^n\lambda\int\limits_{\mathbb{R}}d^{m-n}\lambda\cdot\det_m\left(\begin{array}{c|c}\frac{t(\xi_{p(j)},\lambda_j)}{\sinh(\lambda_j-\xi_{p(k)}-\frac{i\pi}{3})}&\frac{t(\xi_{p(j)},\lambda_j)}{\sinh(\lambda_j-\xi_{p(k)}+\frac{i\pi}{3})}\\\hline\frac{t(\lambda_j,\xi_{p(j)})}{\sinh(\lambda_j-\xi_{p(k)}-\frac{i\pi}{3})}&\frac{t(\lambda_j,\xi_{p(j)})}{\sinh(\lambda_j-\xi_{p(k)}+\frac{i\pi}{3})}\end{array}\right).\end{aligned} \tag{8}$$

Here the sizes of the blocks in the determinant are respectively: $n\times n$; $n\times(m-n)$; $(m-n)\times n$; and $(m-n)\times(m-n)$.

We see that the original m-fold integral over λ_j is now factorized. Indeed, λ_j enters only the j-th line of the determinant and, hence, one can integrate each line separately. Thus, we arrive at

$$\langle Q_\kappa(m)\rangle=\frac{3^m}{2^{m^2}}\prod_{a>b}^{m}\frac{\sinh 3(\xi_a-\xi_b)}{\sinh^3(\xi_a-\xi_b)}\sum_{n=0}^{m}\kappa^{m-n}\sum_{\substack{\{\xi\}=\{\xi_{\gamma_+}\}\cup\{\xi_{\gamma_-}\}\\ |\gamma_+|=n}}\det_m\hat{\Phi}^{(n)}(\{\xi_{\gamma_+}\},\{\xi_{\gamma_-}\})$$

$$\times\prod_{a\in\gamma_+}\prod_{b\in\gamma_-}\frac{\sinh(\xi_b-\xi_a-\frac{i\pi}{3})\sinh(\xi_a-\xi_b)}{\sinh^2(\xi_b-\xi_a+\frac{i\pi}{3})},\quad(9)$$

with

$$\hat{\Phi}^{(n)}(\{\xi_{\gamma_+}\},\{\xi_{\gamma_-}\})=\left(\begin{array}{c|c}\Phi(\xi_j-\xi_k) & \Phi(\xi_j-\xi_k-\frac{i\pi}{3})\\ \hline \Phi(\xi_j-\xi_k+\frac{i\pi}{3}) & \Phi(\xi_j-\xi_k)\end{array}\right),\qquad \Phi(x)=\frac{\sinh\frac{x}{2}}{\sinh\frac{3x}{2}}.\qquad(10)$$

Here the sum is taken with respect to all partitions of the set $\{\xi\}$ into two disjoint subsets $\{\xi_{\gamma_+}\}\cup\{\xi_{\gamma_-}\}$ of cardinality n and $m-n$ respectively. The first n lines and columns of the matrix $\hat{\Phi}^{(n)}$ are associated with the parameters $\xi\in\{\xi_{\gamma_+}\}$. The remaining lines and columns are associated with $\xi\in\{\xi_{\gamma_-}\}$.

Thus, we have obtained an explicit answer for the generating function $\langle Q_\kappa(m)\rangle$ of the inhomogeneous XXZ model. However, the homogeneous limit of (9) is not obvious, since each term of the sum in (9) becomes singular in the limit[1] $\xi_k\to 0$, $k=1,\dots,m$. In order to check that the sum over partitions in the r.h.s. of (9) remains indeed finite in the limit $\xi_k\to 0$, one can introduce again a set of auxiliary contour integrals:

$$\langle Q_\kappa(m)\rangle=\frac{(-1)^{\frac{m^2-m}{2}}3^m}{2^{m^2}m!}\prod_{a>b}^{m}\frac{\sinh 3(\xi_a-\xi_b)}{\sinh(\xi_a-\xi_b)}\sum_{n=0}^{m}\kappa^{m-n}C_m^n\oint\limits_{\Gamma\{\xi-\frac{i\pi}{6}\}}\frac{d^nz}{(2\pi i)^n}\oint\limits_{\Gamma\{\xi+\frac{i\pi}{6}\}}\frac{d^{m-n}z}{(2\pi i)^{m-n}}$$

$$\times\prod_{b=1}^{m}\left\{\prod_{j=1}^{n}\frac{1}{\sinh(z_j-\xi_b+\frac{i\pi}{6})}\prod_{j=n+1}^{m}\frac{1}{\sinh(z_j-\xi_b-\frac{i\pi}{6})}\right\}$$

$$\times\prod_{a=1}^{n}\prod_{b=n+1}^{m}\frac{\sinh(z_a-z_b-\frac{i\pi}{3})\sinh(z_a-z_b+\frac{i\pi}{3})}{\sinh^2(z_a-z_b)}\cdot\det_m\Phi(z_j-z_k).\quad(11)$$

[1]Since (9) depends only on the difference $\xi_j-\xi_k$, one can obtain the homogeneous XXZ chain in the limit $\xi_k\to 0$ as well as in the limit $\xi_k\to -i\zeta/2$.

Here the integration contours $\Gamma\{\xi \mp \frac{i\pi}{6}\}$ surround the points $\{\xi - \frac{i\pi}{6}\}$ for $z_1, \ldots, z_n$ and $\{\xi + \frac{i\pi}{6}\}$ for $z_{n+1}, \ldots, z_m$ respectively. It is easy to check that, taking the residues of the integrand in the simple poles $\{\xi \pm \frac{i\pi}{6}\}$, we immediately reproduce (9). On the other hand, one should simply set $\xi_k = 0$, $k = 1, \ldots, m$ to proceed to the homogeneous limit in (11). As a result we obtain poles of order m in the r.h.s. of (11).

Certainly, the remaining integral is of Cauchy type and, after the change of variables $x_j = e^{2z_j}$, it reduces to the derivatives of order $m-1$ with respect to each x_j at $x_1 = \cdots = x_n = e^{\frac{i\pi}{3}}$ and $x_{n+1} = \cdots = x_m = e^{-\frac{i\pi}{3}}$. If the lattice distance m is not too large, the representations (9), (11) can be successfully used to compute $\langle Q_\kappa(m)\rangle$ explicitely. As an example we give below the list of results for $P_m(\kappa) = 2^{m^2}\langle Q_\kappa(m)\rangle$ up to $m = 9$:

$$
\begin{aligned}
&P_1(\kappa) = 1 + \kappa,\\
&P_2(\kappa) = 2 + 12\kappa + 2\kappa^2,\\
&P_3(\kappa) = 7 + 249\kappa + 249\kappa^2 + 7\kappa^3,\\
&P_4(\kappa) = 42 + 10004\kappa + 45444\kappa^2 + 10004\kappa^3 + 42\kappa^4,\\
&P_5(\kappa) = 429 + 738174\kappa + 16038613\kappa^2 + 16038613\kappa^3 + 738174\kappa^4 + 429\kappa^5,\\
&P_6(\kappa) = 7436 + 96289380\kappa + 11424474588\kappa^2 + 45677933928\kappa^3 + 11424474588\kappa^4\\
&+96289380\kappa^5 + 7436\kappa^6,\\
&P_7(\kappa) = 218348 + 21798199390\kappa + 15663567546585\kappa^2 + 265789610746333\kappa^3\\
&+265789610746333\kappa^4 + 15663567546585\kappa^5 + 21798199390\kappa^6 + 218348\kappa^7,\\
&P_8(\kappa) = 10850216 + 8485108350684\kappa + 39461894378292782\kappa^2\\
&+3224112384882251896\kappa^3 + 11919578544950060460\kappa^4 + 3224112384882251896\kappa^5\\
&+39461894378292782\kappa^6 + 8485108350684\kappa^7 + 10850216\kappa^8\\
&P_9(\kappa) = 911835460 + 5649499685353257\kappa + 177662495637443158524\kappa^2\\
&+77990624578576910368767\kappa^3 + 1130757526890914223990168\kappa^4\\
&+1130757526890914223990168\kappa^5 + 77990624578576910368767\kappa^6\\
&+177662495637443158524\kappa^7 + 5649499685353257\kappa^8 + 911835460\kappa^9.
\end{aligned}
\tag{12}
$$

Using the results (12), we can calculate the correlation function $\langle \sigma_1^z \sigma_{m+1}^z \rangle$ up to $m = 8$:

$$\begin{aligned}
\langle \sigma_1^z \sigma_2^z \rangle &= -2^{-1}, \\
\langle \sigma_1^z \sigma_3^z \rangle &= 7 \cdot 2^{-6}, \\
\langle \sigma_1^z \sigma_4^z \rangle &= -401 \cdot 2^{-12}, \\
\langle \sigma_1^z \sigma_5^z \rangle &= 184453 \cdot 2^{-22}, \\
\langle \sigma_1^z \sigma_6^z \rangle &= -95214949 \cdot 2^{-31}, \\
\langle \sigma_1^z \sigma_7^z \rangle &= 1758750082939 \cdot 2^{-46}, \\
\langle \sigma_1^z \sigma_8^z \rangle &= -30283610739677093 \cdot 2^{-60}, \\
\langle \sigma_1^z \sigma_9^z \rangle &= 5020218849740515343761 \cdot 2^{-78}.
\end{aligned} \tag{13}$$

For $m = 2, 3$ we reproduce the results [9], [10] where the two-point correlation functions of the XXZ chain were calculated up to the lattice distance $m = 3$ for general Δ.

One can also compare the exact results (13) with the values given by the asymptotic formula suggested in [11].

m	$\langle \sigma_1^z \sigma_{m+1}^z \rangle$ Exact	$\langle \sigma_1^z \sigma_{m+1}^z \rangle$ Asymptotics
1	-0.5000000000	-0.5805187860
2	0.1093750000	0.1135152692
3	-0.0979003906	-0.0993588501
4	0.0439770222	0.0440682654
5	-0.0443379157	-0.0444087865
6	0.0249933420	0.0249365346
7	-0.0262668452	-0.0262404925
8	0.0166105110	0.0165641239

One can see that the correlation function $\langle \sigma_1^z \sigma_{m+1}^z \rangle$ approaches its asymptotic regime very quickly and that, already for $m = 4$, the relative precision is better than 1%.

In conclusion, we would like to comment on the results obtained in this paper. We have shown that, for the special value of the anisotropy parameter $\Delta = 1/2$, the multiple integral giving the generating function $\langle Q_\kappa(m) \rangle$ can be factorized and computed explicitly in terms of the inhomogeneities. It was shown recently in the papers [12], [13] that a factorization of the multiple integral representations for the correlation functions of the inhomogeneous XXZ chain is possible for generic value of Δ. However, as in [13], we see that it is not straightforward to take the explicit homogeneous limit of (9) for general m, although it is possible to do it for small distances. One way to obtain explicitly this homogeneous limit is to come back to the multiple integral of Cauchy type (11), but with non-factorized integrand.

Concerning the exact numerical formulas we obtained for the generating function $\langle Q_\kappa(m)\rangle$ up to $m = 9$, we would like to mention the remarkable fact that all the coefficients of the polynomials $P_m(\kappa)$ turned out to be integers. We conjecture that this property holds for all m. In particular, the highest and lowest coefficients of $P_m(\kappa)$ correspond to the emptiness formation probability and, hence, are equal to the number of alternating sign matrices of the size $m \times m$ [2], [3]. For the other coefficients, this property should be related to the structure of the ground state at $\Delta = 1/2$ conjectured by Razumov and Stroganov in [2] (see also [14], [15], [16]). It would be very interesting to reveal the combinatorial nature of these coefficients.

Acknowledgements

J. M. M., N. S. and V. T. are supported by CNRS. N. K., J. M. M., V. T. are supported by the European network EUCLID-HPRNC-CT-2002-00325. J. M. M. and N.S. are supported by INTAS-03-51-3350. N.S. is supported by the French-Russian Exchange Program, the Program of RAS Mathematical Methods of the Nonlinear Dynamics, RFBR-05-01-00498, Scientific Schools 2052.2003.1. N. K, N. S. and V. T. would like to thank the Theoretical Physics group of the Laboratory of Physics at ENS Lyon for hospitality, which makes this collaboration possible.

References

[1] W. Heisenberg, Zeitschrift für Physik, **49** (1928) 619.

[2] A. V. Razumov and Y. G. Stroganov, J. Phys. A **34**, (2001) 3185, cond-mat/0012141.

[3] N. Kitanine, J. M. Maillet, N. A. Slavnov, V. Terras, J. Phys. A: Math. Gen. **35** (2002) L753, hep-th/0201134.

[4] M. Jimbo, K. Miki, T. Miwa and A. Nakayashiki, Phys. Lett. A **168** (1992) 256.

[5] M. Jimbo and T. Miwa, Journ. Phys. A: Math. Gen., **29** (1996) 2923.

[6] N. Kitanine, J. M. Maillet and V. Terras, Nucl. Phys. B, **567** [FS] 554. (2000), math-ph/9907019.

[7] N. Kitanine, J. M. Maillet, N. A. Slavnov, V. Terras, Nucl. Phys. B **641** [FS] (2002) 487, hep-th/0201045.

[8] A. G. Izergin and V. E. Korepin, Commun. Math. Phys. **99** (1985) 271.

[9] M. Shiroishi, M. Takahashi, and K. Sakai, J. Phys. A **36**, (2003) L337, cond-mat/0304475.

[10] M. Shiroishi, M. Takahashi, and K. Sakai, J. Phys. A **37**, (2003) 5097, cond-mat/0402625.

[11] S. Lukyanov, Phys. Rev. **B59** (1999) 11163, hep-th/9809254.

[12] H. Boos, M. Jimbo, T. Miwa, F. Smirnov, and Y. Takeyama, A recursion formula for the correlation functions of an inhomogeneous XXX model, hep-th/0405044.

[13] H. Boos, M. Jimbo, T. Miwa, F. Smirnov, and Y. Takeyama, Reduced qKZ equation and correlation functions of the XXZ model, hep-th/0412191.

[14] A. V. Razumov and Y. G. Stroganov, Theor. Math. Phys. **138** (2004) 333, math.CO/0104216.

[15] J. de Gier, M. T. Batchelor, B. Nienhuis, S. Mitra, J. Math. Phys. **43** (2002) 4235, math-ph/0110011.

[16] S. Mitra, B. Nienhuis, J. de Gier, M. T. Batchelor, JSTAT (2004) P09010, cond-mat/0401245.